SOME COMMENTS ON CATEGORIES, PARTICLES AND INTERACTIONS

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Abstract

An algebraic formalism for the study of interacting particle systems is considered. Particle processes are described in terms of the category theory. The problem for the unique description of these processes is discussed. Categories relevant for this subject are described. The concept of generalized transmutations of interacting particle systems is introduced. The connection with a system with some generalized statistics is explained.

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1 Introduction

It is well–known that particles like baryons, nuclei, atoms or molecules are characterized by their own specific excitation spectrum. The existence of these spectra is one of the fundamental properties of the structure of matter. Suppose that we have a particle system with a collection of bound–state energy levels. There is a ground state and there are several excited states. It is also well–known that there are some transitions between different levels. They are results of interactions of the system with an external field or with other particle systems. These transitions which need more energy are known as excitation processes. They are the result of absorption of quanta of an external field. On the other hand there are transitions connected with an energy spending, they correspond to some decay processes. It is interesting that there are also processes with no energy change. Let us consider for instance these processes which can be described as sequences of vertex interactions of particle charges with an external quantum field. Charged particles are transformed under these interactions into a composite nonlocal discrete system which contains charges and quanta of the field. These systems are said to be *dressed particles* [1]. We describe the structure of these dressed particles as a lattice with n sites, $n = 1, 2, \ldots$. Every lattice site is a center for a vertex interaction of charge with the external quantum field. We assume that there are N elementary excitation states on every lattice sites. There are also collective excitation states. It should be interesting to consider the general formalism for the study of all possible collective excitations of such system. We can imagine our lattice as a d-dimensional space (a manifold) equipped with n distinct points as lattice sites. One can consider a quantum dot, spin chains, or a set of vertex interactions of particles moving in two–dimensional space under influence of transversal magnetic field as examples of such systems corresponding for $d = 0, 1, 2$, respectively. Our fundamental assumptions are that a collection of

- initial configurations of the system,
- elementary particle processes.

is given. It seems to be natural to assume that every possible configurations of the system can be obtained as a result of certain physical processes. We also assume that every process can be described as a sequence of elementary ones. These elementary processes represent elementary acts of lattice interactions. If all final configurations for the system under consideration can be

described in an unique way as a result of transformation of an initial configuration, then we say that the system is equipped with a *category symmetry* or *coherent evolution*. Every such transformation is said to be a *evolution transformation*. This means that our category symmetry is in fact a formalism for the description of particle interactions. The problem is to determine for a given system the smallest collection of symmetry transformations generating all others in an unique way.

The classical notion of the concept of symmetry in physics is based on group theory. The role played by the group representation theory for the study of symmetries in particle physics is well–known. The construction of a tensor product of representations is essential for such study. For instance, it allows to built states for composite systems of particles from single particles ones. The unitary symmetry and corresponding quark model is here a good example. It is known that we need a comultiplication for a tensor product of representations. It is interesting that a comultiplication does exist for a large class of q–deformed universal enveloping algebras. Hence they provide new possibilities for the study of particles, fields and their interactions in mathematical physics. Also categories which contain a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ called a monoidal operation [2, 3, 4] provide some additional possibilities. Such categories are said to be monoidal. They can contain more structures and operations. The name "monoidal" indicate, that there is one essential operation, just the monoidal one, other operations play an auxiliary role. The bifunctor $\otimes$ plays a similar role like the usual tensor product of group representations.

Note that a related subject has been studied previously by several authors [5, 6, 7] and others. Formalism corresponding to the braided symmetry has been developed mainly by Majid [8, 9, 10, 11, 12, 13]. Categories in the context of quantum groups has been presented by Kassel [14]. The application of categories in the topological quantum field theory has been considered by Sawin [15]. Similar formalism has been also developed previously by the author [1]. Note that all these studies can be included in our general scheme. One can also consider the q–extended supersymmetry concept [16, 17] as a particular example of our general formalism. Our considerations are mainly motivated by applications for the investigation of interacting systems of particles in low–dimensional spaces, but there are more different possible applications. It is known that the study of certain integrable models on a lattice leads to the investigation of some new formalism [18]. Hence it is interesting

to study all these additional possibilities for the developing of the formalism beyond of the quantum mechanics and field theory.

In this paper we are going to study these additional possibilities in a general manner in terms of monoidal categories. All our considerations are on abstract algebraic level. We would like to consider the most fundamental algebraic structures suitable for the description of particle interactions. We would like also to discuss the physical application for the classification of interacting particle systems. One can further develop our concept in terms of quantum von–Neumann algebras and their representations [19]. The paper is organized as follows. In Section 2 the general concept of particle interactions is considered in terms of monoidal categories. Particle processes are described as certain transformations of categories. The essential problem is the unique description, it is related to the coherence in categories [2]. In Section 3 categories relevant for our goal are described in details. Commutation relations for creation and annihilation quantum processes are described as certain specific transformations. They lead to the system with generalized statistics [20]. An introduction to the category theory is given in the Appendix. We believe that our approach can be useful for the deeper understanding of such new methods in quantum optics or both condensed matter and particle physics.

2 General considerations

Let us consider a system of hard core particles moving on certain $d+1$–dimensional space–time manifold under influence of some external field. All our considerations are based on the assumption that there is the vacuum state $\mathbf{1} \equiv |0\rangle$, the lowest energy elementary excited states $\{|i\rangle\}_{i=1}^{N}$, and their conjugated states $\mathbf{1}^* \equiv \langle 0|$, and $\{\langle i|\}_{i=N}^{1}$ with the scalar product $\langle i|j\rangle \in \mathbb{C}$. We also assume that there are collective excitations of the system which can be described as a result of certain multiple product of elementary excitations. There is an energy gap between the vacuum state $\mathbf{1} \equiv |0\rangle$ and the lowest energy excited state $|i\rangle$. A finite set of N operators $L = \{x^i\}_{i=1}^{N}$ is given as a starting point for our considerations. Every such operator act on the Hilbert space of functions on d–dimensional space. We also assume that these operators transform functions representing the ground state of the system into states representing elementary excitations of the system, i.e. $|i\rangle := x^i|0\rangle$.

In this way these operators represent elementary excited states of our system. Hence these operators are said to be primary. For the description of other states representing for instance collective excitations we need a product of operators. Such product need not forms a closed algebra but it must be defined in an unique way. A product of n arbitrary primary operators should represents a collective n–tuple excitation. It is an analog of a n particle state. We would like to study such product in terms of the category theory. If x^i is a primary operator, then there is a corresponding vector space $\mathcal{U} = \mathcal{U}(x^i)$. It is formally a $\mathbb{C}$–linear span of x^i, i. e. $\mathcal{U}(x^i) := \{\alpha x^i; \alpha \in \mathbb{C}\}$. The $\mathbb{C}$–linear span of the ground state $\mathbf{1}$ is denoted by I. It said to be the unit object. If $\mathcal{U}$ and $\mathcal{V}$ are $\mathbb{C}$–linear spans of x^i and x^j, respectively, then the linear span corresponding for certain product of these operators is denoted by $\mathcal{U} \otimes \mathcal{V}$ and is also said to be a product of $\mathcal{U}$ and $\mathcal{V}$. If for example $\mathcal{U}$ represents charged particles excitation and $\mathcal{V}$ some quanta, then the product $\mathcal{U} \otimes \mathcal{V}$ describes the composite system containing both particles and quanta. This means that the operation $\otimes : \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{U} \otimes \mathcal{V}$ describes the "composition" process of states. Such process tell us how to built a space of composite quantum states of the system from elementary ones. Hence it can be also understood as a generalization of the usual tensor product of group representations. Observe that the arrow $\mathcal{U} \longrightarrow \mathcal{U} \otimes \mathcal{V}$ describes the process of absorption and the arrow $\mathcal{U} \otimes \mathcal{V} \longrightarrow \mathcal{U}$ describes the process of emission.

Let us denote by $\mathbb{P}$ the collection of all formal linear spans of primary operators, i.e. $\mathbb{P} := \{\mathcal{U} = \mathcal{U}(x^i) : i = 1, \cdots, N\}$. The collection of complex conjugated spaces is denoted by $\mathbb{P}^*$. We assume that an arbitrary sequence consists of the unit object I or spaces from the collection $\mathbb{P}$ or $\mathbb{P}^*$ represents initial configuration of our system. These configurations can be transformed into some new ones by a set of certain transformations. These transformations represent certain physical processes like composition, emission, absorption, etc... It is obvious that these transformations can be coherent or not. Coherence for a set of transformations means path–independent construction of these transformations. Note that the coherence problem can be expressed graphically in terms of tangle tree operads [25]. Our goal is the construction of a collection of transformations which transform in an unique way initial configurations into final ones – representing the result of interactions. We denote $\mathcal{E}$ the generating set for these transformations. Let us consider some examples.

Example 1. If $\mathcal{E}$ contains only one operation, namely the product $\otimes$:

$\mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P} \otimes \mathbb{P}$, then starting from this product we can construct a set of multiproducts $\otimes^n : \mathbb{P}^{\times n} \longrightarrow \mathbb{P}^{\otimes^n}$ such that $\otimes^2 \equiv \otimes$ and every multiproduct $\otimes^n$ for $n > 2$ can be calculated by an iteration procedure. Such procedure need not be unique. For instance for $n = 3$ we obtain $\otimes^3 := \otimes \circ (\otimes \times id)$, But we also obtain $\otimes^3 := \otimes \circ (id \times \otimes)$. Hence for the uniqueness we need some additional assumptions like the associativity constraints, see the Appendix for more details.

Example 2. For the (left) $*$–operation we use the standard relations

$$\mathcal{U}^{**} = \mathcal{U}, \quad (\mathcal{U} \otimes \mathcal{V})^* = \mathcal{V}^* \otimes \mathcal{U}^*. \tag{1}$$

In this case $\mathcal{E} := \{\otimes.*\}$.

Example 3. We introduce a generating set $g(\mathbb{P}) := \{g_{\mathcal{U}} : \mathcal{U} \in \mathbb{P}\}$ of I–valued mappings $g_{\mathcal{U}} : \mathcal{U}^* \otimes \mathcal{U} \longrightarrow I$, where

$$g_{\mathcal{U}}(x^{*i} \otimes x^j) \equiv (x^{*i}|x^j) := \langle i|j\rangle \tag{2}$$

for pairing g. The extension $g_{\mathcal{U}\otimes\mathcal{V}}$ of this pairing to the product $\mathcal{U} \otimes \mathcal{V}$ is a problem. We need here the following commutative diagram

$$\begin{array}{ccc} \mathcal{V}^* \otimes \mathcal{U}^* \otimes \mathcal{U} \otimes \mathcal{V} & \overset{id_{\mathcal{V}^*} \otimes g_{\mathcal{U}} \otimes id_{\mathcal{V}}}{\longrightarrow} & \mathcal{V}^* \otimes \mathcal{V} \\ \| & & \downarrow g_{\mathcal{V}} \\ (\mathcal{U} \otimes \mathcal{V})^* \otimes (\mathcal{U} \otimes \mathcal{V}) & \underset{g_{\mathcal{U}\otimes\mathcal{V}}}{\longrightarrow} & I \end{array} \tag{3}$$

for the extension. We can introduce the set $g(\mathbb{P}^*) := \{g_{\mathcal{U}} : \mathcal{U} \in \mathbb{P}\}$ of I–valued mappings $g_{\mathcal{U}^*} : \mathcal{U} \otimes \mathcal{U}^* \longrightarrow I$ in a similar way. For the extension we use the diagram

$$\begin{array}{ccc} \mathcal{U} \otimes \mathcal{V} \otimes \mathcal{V}^* \otimes \mathcal{U}^* & \overset{id_{\mathcal{U}} \otimes g_{\mathcal{V}^*} \otimes id_{\mathcal{U}^*}}{\longrightarrow} & \mathcal{U} \otimes \mathcal{U}^* \\ \| & & \downarrow g_{\mathcal{U}^*} \\ (\mathcal{U} \otimes \mathcal{V}) \otimes (\mathcal{U} \otimes \mathcal{V})^* & \underset{g_{(\mathcal{U}\otimes\mathcal{V})^*}}{\longrightarrow} & I \end{array} \tag{4}$$

In the way we can construct a category $\mathcal{M} := \mathcal{M}(\mathbb{P}, \mathcal{E})$ whose objects are multiple products of object of $\mathbb{P}$ and morphisms are obtained by iteration procedures applied to operations from the set $\mathcal{E}$. The monoidal operation of $\mathcal{M}$ can also be obtained by the proper iteration of the initial product $\otimes$.

Note that there is the uniqueness problem with the construction of the category $\mathcal{M} = \mathcal{M}(\mathbb{P}, \mathcal{E})$ over $\mathbb{P}$. One can construct many different categories $\mathcal{M}$ for a given initial collection of spaces $\mathbb{P}$ with different generating set $\mathcal{E}$. We denote by $\mathbf{Cat}(\mathbb{P})$ the class of all these categories. Let $\mathcal{M}(\mathbb{P}, \mathcal{E})$ and $\mathcal{N}(\mathbb{P}, \mathcal{E}')$ be two such categories. An arbitrary functor $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$ which transform the set of operations $\mathcal{E}$ into $\mathcal{E}'$ is said to be a *generalized transmutation.* In this case we say that the set $\mathcal{E}$ is transmuted into $\mathcal{E}'$. The category $\mathcal{N}$ is then said to be *functored* over $\mathcal{M}$ [31].

If $\mathcal{E} := \{I, \otimes\}$ and $\mathcal{E}' := \{I', \underline{\otimes}\}$, then the corresponding generalized transmutation $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$ is a monoidal functor of categories $\mathcal{M}$ and $\mathcal{N}$. This means that it is a triple

$$\mathcal{F} := \{\mathcal{F}, \varphi_2, \varphi_0\} : \mathcal{M} \longrightarrow \mathcal{N}$$

which consists of a functor $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$, a natural isomorphism

$$\varphi := \varphi_{2,\mathcal{U},\mathcal{V}} : \mathcal{F}\mathcal{U}\underline{\otimes}\mathcal{F}\mathcal{V} \longrightarrow \mathcal{F}(\mathcal{U} \otimes \mathcal{V}),$$

and an isomorphism $\varphi_0 : I \longrightarrow \mathcal{F}I = I'$, such that the following diagrams

$$\begin{array}{ccc} \mathcal{F}\mathcal{U}\underline{\otimes}\mathcal{F}\mathcal{V}\underline{\otimes}\mathcal{F}\mathcal{V} & \xrightarrow{\varphi_2\underline{\otimes}id} & \mathcal{F}(\mathcal{U} \otimes \mathcal{V})\underline{\otimes}\mathcal{F}\mathcal{W} \\ id \otimes \varphi_2 \downarrow & & \downarrow \varphi_2 \\ \mathcal{F}\mathcal{U}\underline{\otimes}\mathcal{F}(\mathcal{V} \otimes \mathcal{W}) & \xrightarrow[\varphi_2]{} & \mathcal{F}(\mathcal{U} \otimes \mathcal{V} \otimes \mathcal{W}) \end{array} \tag{5}$$

$$\begin{array}{ccc} \mathcal{F}I\underline{\otimes}\mathcal{F}\mathcal{U} & \xrightarrow{\varphi_2} & \mathcal{F}(I \otimes \mathcal{U}) \\ \varphi_0\underline{\otimes}id \uparrow & \swarrow & \mathcal{F}(l_\mathcal{U}) \\ \mathcal{F}\mathcal{U} & & \end{array} \tag{6}$$

$$
\begin{tikzcd}
\mathcal{F}\mathcal{U}\underline{\otimes}\mathcal{F}I \arrow[r, "\varphi_2"] & \mathcal{F}(\mathcal{U}\otimes I) \arrow[dl, "\mathcal{F}(r_{\mathcal{U}})"] \\
\mathcal{F}\mathcal{U} \arrow[u, "id\underline{\otimes}\varphi_0"] &
\end{tikzcd}
\tag{7}
$$

are commutative. If φ_2 and φ_0 are identities, then $\mathcal{F}$ is said to be strict. For the $*$–operation and pairing we have

$$
(\mathcal{F}(\mathcal{U}))^* = \mathcal{F}(\mathcal{U}^*), \quad g_{\mathcal{U}} = g'_{\mathcal{F}(\mathcal{U})}; \tag{8}
$$

respectively. A generalized transmutation $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$ is said to be cross symmetric if the following diagram

$$
\begin{tikzcd}
\mathcal{F}\mathcal{U}^*\underline{\otimes}\mathcal{F}\mathcal{V} \arrow[r, "\varphi_2"] \arrow[d, "\Psi'"'] & \mathcal{F}(\mathcal{U}^*\otimes\mathcal{V}) \arrow[d, "\mathcal{F}(\Psi)"] \\
\mathcal{F}\mathcal{V}\underline{\otimes}\mathcal{F}\mathcal{U}^* \arrow[r, "\varphi_2"'] & \mathcal{F}(\mathcal{V}\otimes\mathcal{U}^*)
\end{tikzcd}
\tag{9}
$$

is commutative for every generating objects $\mathcal{U}, \mathcal{V}$ of $\mathcal{M}$, where Ψ and Ψ' are the cross symmetries in $\mathcal{M}$ and $\mathcal{N}$, respectively. In the case of a braid symmetries we have the following diagram

$$
\begin{tikzcd}
\mathcal{F}\mathcal{U}\underline{\otimes}\mathcal{F}\mathcal{V} \arrow[r, "\varphi_2"] \arrow[d, "\Psi'"'] & \mathcal{F}(\mathcal{U}\otimes\mathcal{V}) \arrow[d, "\mathcal{F}(\Psi)"] \\
\mathcal{F}\mathcal{V}\underline{\otimes}\mathcal{F}\mathcal{U} \arrow[r, "\varphi_2"'] & \mathcal{F}(\mathcal{V}\otimes\mathcal{U})
\end{tikzcd}
\tag{10}
$$

Note that the functor $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$ generalizes the Majid concept of transmutation of braid statistics [32]. If $\mathcal{M}$ is the category of cobordisms of smooth manifolds and $\mathcal{N}$ is a category of vector spaces, that the generalized

transmutation is known as the Topological Quantum Field Theory [15].
Example 4. Let H and H' be two Hopf algebras. If $h : H \longrightarrow H'$ is a Hopf algebra homomorphism, then we can introduce the transmutation $\mathcal{F} : \mathcal{M}^H \longrightarrow \mathcal{M}^{H'}$ of categories of right comodules as follows. The functor $\mathcal{F}$ acts as the identity functor on arbitrary comodule $\mathcal{U}$ but the coaction $\rho_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathcal{U} \otimes H$ transform into new one, namely into $\rho'_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathcal{U} \otimes H'$, where

$$\rho'_{\mathcal{U}} := (id_{\mathcal{U}} \otimes h) \circ \rho_{\mathcal{U}}. \tag{11}$$

For coquasitriangular Hopf algebras H and H' with coquasitriangular structures $\langle -, - \rangle : H \otimes H \longrightarrow I$ and $\langle -, - \rangle' : H' \otimes H' \longrightarrow I$, respectively, we obtain

$$\Psi'_{\mathcal{U},\mathcal{V}}(u \otimes v) = \Sigma \langle h(v_1), h(u_1) \rangle' \, v_0 \otimes u_0, \tag{12}$$

where $\rho(u) = \Sigma u_0 \otimes u_1 \in \mathcal{U} \otimes H$, $\rho(v) = \Sigma v_0 \otimes v_1 \in \mathcal{V} \otimes H$ for every $u \in \mathcal{U}, v \in \mathcal{V}$, and $\langle k, l \rangle = \langle h(k), h(l) \rangle'$ for every $k, l \in H$.
Example 5. Let $H := \mathbb{C}G$ and $H' := \mathbb{C}G'$ be group algebras, where G and G' are Abelian groups equipped with factors ϵ and ϵ', respectively. Then the transmutation : $\mathcal{M}(\mathsf{G}, \epsilon) \longrightarrow \mathcal{M}(\mathsf{G}', \epsilon')$ is determined by a group homomorphism $h : G \longrightarrow G'$ such that

$$\epsilon(\alpha, \beta) = \epsilon(h(\alpha), h(\beta)) \tag{13}$$

for $\alpha, \beta \in G$.

3 Commutation relations

Let us denote by $\mathbb{P}(n)$ the collection of all tensor products of the form

$$\mathcal{U}_{i_1} \otimes \cdots \otimes \mathcal{U}_{i_n}, \tag{14}$$

for all $\mathcal{U}_{i_1}, \cdots, \mathcal{U}_{i_n} \in \mathbb{P}$. We introduce $\mathbb{P}^*(n)$ in a similar way. We also introduce the collection $(\mathbb{P}^* \otimes \mathbb{P})(n)$ of sequences of the form

$$\mathcal{U}^*_{j_n} \otimes \cdots \otimes \mathcal{U}^*_{j_1} \otimes \mathcal{U}_{i_1} \otimes \cdots \otimes \mathcal{U}_{i_n}. \tag{15}$$

The collection $(\mathbb{P} \otimes \mathbb{P}^*)(n)$ can be defined in an obvious way. We have the following examples.
Example 4. Let us denote by $\mathcal{M} := \mathcal{M}(\mathbb{P}, I, \otimes, *, g)$ the category $\mathcal{M} :=$

$\mathcal{M}(\mathbb{P},\mathcal{E})$, where $\mathcal{E} := \{I, \otimes, *, g\}$. We introduce two sets of transformations $a^+ := \{a^+_{\mathcal{U}} : \mathbb{P}(n) \longrightarrow \mathbb{P}(n+1)\}$ and $a^- := \{a^-_{\mathcal{U}^*} : \mathbb{P}(n) \longrightarrow \mathbb{P}(n-1)\}$, where

$$a^+_{\mathcal{U}}(\mathcal{U}_{i_1} \otimes \cdots \otimes \mathcal{U}_{i_n}) := \mathcal{U} \otimes \mathcal{U}_{i_1} \otimes \cdots \otimes \mathcal{U}_{i_n}, \tag{16}$$

and

$$a^-_{\mathcal{U}^*}(\mathcal{U}_{i_1} \otimes \cdots \otimes \mathcal{U}_{i_n}) := g_{\mathcal{U}}(\mathcal{U}^*_{j_1} \otimes \mathcal{U}_{i_1}) \otimes \cdots \otimes \mathcal{U}_{i_n} \tag{17}$$

where

$$g_{\mathcal{U}}(\mathcal{U}^*_{j_1} \otimes \mathcal{U}_{i_1}) \begin{cases} g_{\mathcal{U}} & \text{for} \quad \mathcal{U}^*_{j_i} \equiv \mathcal{U}^*, \mathcal{U}_{i_1} \equiv \mathcal{U} \\ 0 & \text{otherwise} \end{cases} , \tag{18}$$

for $\mathcal{U}_{i_1}, \cdots, \mathcal{U}_{i_n}, \mathcal{U}$ and $\mathcal{U}^*_{j_1}, \mathcal{U}^* \in \mathbb{P}$. Here $\mathcal{U}^*$ represents a quasihole and $\mathcal{U}$ – a quasiparticle. Two different objects $\mathcal{U}, \mathcal{V} \in \mathbb{P}$ represent (quasi-) particle states of two different sorts. There are no identical particles. It is easy to see that we have the following set of relations

$$a^-_{\mathcal{U}^*} \circ a^+_{\mathcal{U}} = g_{\mathcal{U}} \mathbf{1}, \tag{19}$$

where $\mathcal{U} \in \mathbb{P}$. These relations are in fact the commutation relations for the system equipped with the infinite statistics [23]. One can use the relation

$$\hat{a}_{\mathcal{U}} \circ \hat{a}_{\mathcal{U}} = \hat{a}_{\mathcal{U} \otimes \mathcal{V}},$$

where $\hat{a}_{\mathcal{U}}$ stands for $a^+_{\mathcal{U}}$ or $a_{\mathcal{U}^*}$, for the extension of commutation relations corresponding for monoidal products of generating objects. These relations seems to be simple, but they lead to well–defined operator algebras [33]. Observe that we have here elementary quantum processes of two sorts, namely creation and annihilation.

Example 5. We assume that $\mathcal{E}$ contains the cross symmetry Ψ_{cross} in addition to the previous example. The corresponding category is denoted by $\mathcal{M}_{cross} := \mathcal{M}(\mathbb{P}, I, \otimes, *, g, \Psi_{cross})$. We need here a collection $\Psi(\mathbb{P}) := \{\Psi_{\mathcal{U}^*,\mathcal{V}} : \mathcal{U}^*, \mathcal{V} \in \mathbb{P}\}$ as the initial data for the description of exchange processes of (quasi-) particles and (quasi-) holes, see the Appendix. We have here the following relations

$$b^-_{\mathcal{U}^*} \circ b^+_{\mathcal{U}} - \ b^+_{\mathcal{U}} \circ b^-_{\mathcal{U}^*} \circ \Psi_{\mathcal{U}^*,\mathcal{U}} := g_{\mathcal{U}} \mathbf{1}, \tag{20}$$

where

$$\begin{aligned} &b^+_{\mathcal{U}} := a^+_{\mathcal{U}}, \\ &b^-_{\mathcal{U}^*}(\mathcal{V}_1 \otimes \mathcal{V}_2) := \\ &[(a^- \otimes id_{\mathcal{V}_2}) - (id_{\mathcal{V}_1} \otimes a^-) \circ (\Psi_{\mathcal{U}^*,\mathcal{V}} \otimes id_{\mathcal{V}_2})]\,(U^* \otimes \mathcal{V}_1 \otimes \mathcal{V}_2), \end{aligned} \tag{21}$$

and

$$a^{-}(\mathcal{U}^{*}\otimes\mathcal{V}) := a^{-}_{\mathcal{U}^{*}}(\mathcal{V}). \tag{22}$$

In this way we have here a collection of elementary processes $\mathcal{E}$ which contains creation, annihilation and exchange processes.

Example 6. We replace the cross symmetry by the braid one. In this case we obtain the following relations

$$c^{-}_{\mathcal{U}^{*}}\circ c^{+}_{\mathcal{U}} - \ c^{+}_{\mathcal{U}}\circ c^{-}_{\mathcal{U}^{*}}\circ\Psi_{\mathcal{U}^{*},\mathcal{U}} := g_{\mathcal{U}}\mathbf{1}, \tag{23}$$

and in addition

$$\begin{array}{l} c^{-}_{\mathcal{U}^{*}}\circ c^{+}_{\mathcal{V}} - \ c^{+}_{\mathcal{V}}\circ c^{-}_{\mathcal{U}^{*}}\circ\Psi_{\mathcal{U}^{*},\mathcal{V}} = 0,\\ c^{+}_{\mathcal{U}}\circ c^{+}_{\mathcal{V}} - \ c^{+}_{\mathcal{V}}\circ c^{+}_{\mathcal{U}}\circ\Psi_{\mathcal{U},\mathcal{V}} = 0,\\ c^{-}_{\mathcal{U}^{*}}\circ c^{-}_{\mathcal{V}^{*}} - \ c^{-}_{\mathcal{V}^{*}}\circ c^{-}_{\mathcal{U}^{*}}\circ\Psi_{\mathcal{U}^{*},\mathcal{V}^{*}} = 0. \end{array} \tag{24}$$

Note that for the braid symmetry there are additional elementary quantum processes, namely the exchange processes of identical particles on lattice in two dimensional case.

Example 7. Let $\mathcal{M}_{cross} := \mathcal{M}(\mathbb{P}, I, \otimes, *, g, \Psi_{cross})$ be a category with commutation relation1s like in the Example 5. we denote by $\mathcal{N}_{cross} := \mathcal{N}(\mathbb{P}, I, \underline{\otimes},$-$*, g', \Psi'_{cross})$ a second category with a new cross Ψ'_{cross} and pairing g'. One can define the following two sets $c^{+} := \{c^{+}_{\mathcal{U}} : \mathbb{P}(n) \longrightarrow \mathbb{P}(n+1)\}$ and $c^{-} := \{c^{-}_{\mathcal{U}^{*}} : \mathbb{P}(n) \longrightarrow \mathbb{P}(n-1)\}$ of operators in it. For a cross symmetric generalized transmutation $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$, we have the relation for these operators

$$c^{-}_{\mathcal{F}(\mathcal{U}^{*})}\circ c^{+}_{\mathcal{F}(\mathcal{U})} - \ c^{+}_{\mathcal{F}(\mathcal{U})}\circ c^{-}_{\mathcal{F}(\mathcal{U}^{*})}\circ\Psi'_{\mathcal{F}(\mathcal{U}^{*}),\mathcal{F}(\mathcal{U})} := g'_{\mathcal{F}(\mathcal{U})}\mathbf{1}, \tag{25}$$

Note that the category $\mathcal{N}$ can be braided or symmetric. In these cases we obtain additional relations such as (25).

It is obvious that the concept of category symmetries is related to the systems with generalized statistics [20, 23]. Note that the braid commutation relations, consistency conditions and corresponding Fock space representation with well–defined scalar product has been considered previously, see [30] for instance. Some interesting examples of related formalism has been studied previously by Fiore [34]. Observe that the above concept of category symmetries can be futher developed in a few respects. One can consider the corresponding noncommmutative calculi, It should be interesting to study Hamiltonians in terms of described here creation and annihilation operators and study the concrete physical models.

Appendix

Let us briefly recall the fundamental concept of the category theory for the fixing of notation. For more details see the textbook of Mac Lane [2]. A category $\mathcal{M}$ contains a collection $\mathcal{O}b(\mathcal{M})$ of objects and a collection $hom(\mathcal{M})$ of arrows (morphisms). The collection $hom(\mathcal{M})$ is the union of mutually disjoint sets $hom(\mathcal{U},\mathcal{V})$ of arrows $f:\mathcal{U}\longrightarrow\mathcal{V}$ from $\mathcal{U}$ to $\mathcal{V}$ defined for every pair of objects $\mathcal{U},\mathcal{V}\in\mathcal{O}b(\mathcal{M})$. It may happen that for a pair $\mathcal{U},\mathcal{V}\in\mathcal{O}b(\mathcal{M})$ the set $hom(\mathcal{U},\mathcal{V})$ is empty. The associative composition of morphisms is also defined. A functor $\mathcal{F}:\mathcal{M}\longrightarrow\mathcal{N}$ of the category $\mathcal{M}$ into the category $\mathcal{N}$ is a map which sends objects of $\mathcal{M}$ into objects of $\mathcal{N}$ and morphisms of $\mathcal{M}$ into morphisms of $\mathcal{N}$ such that $\mathcal{F}(f\circ g)=\mathcal{F}(f)\circ\mathcal{F}(g)$ for every morphisms $f:\mathcal{V}\longrightarrow\mathcal{W}$ and $g:\mathcal{U}\longrightarrow\mathcal{V}$ of $\mathcal{M}$. The generalization to multifunctors is obvious. One can consider an arbitrary object of a category as an example of constant functor. For instance an n–ary functor $\mathcal{F}:\mathcal{M}^{\times n}\longrightarrow\mathcal{N}$ sends an n–tuple of objects of $\mathcal{M}$ into an object of $\mathcal{N}$. The corresponding condition for morphisms is evident. In this paper we restrict our attention for a description how functors act on objects, we omit the action on morphisms for simplicity. The reader can complete our description.

Now we recall the concept of natural transformations. Let $\mathcal{F}$ and $\mathcal{G}$ be two functors of the category $\mathcal{M}$ into the category $\mathcal{N}$. A natural transformation $s:\mathcal{F}\longrightarrow\mathcal{G}$ of $\mathcal{F}$ into $\mathcal{G}$ is a collection of morphisms $s=\{s_{\mathcal{U}}:\mathcal{F}(\mathcal{U})\longrightarrow\mathcal{G}(\mathcal{U}),\mathcal{U}\in\mathcal{O}b(\mathcal{M})\}$ such that

$$s_{\mathcal{V}}\circ\mathcal{F}(f)=\mathcal{G}(f)\circ s_{\mathcal{U}} \tag{26}$$

for every morphism $f:\mathcal{U}\longrightarrow\mathcal{V}$ of $\mathcal{M}$. The set of all natural transformations of $\mathcal{F}$ into $\mathcal{G}$ is denoted by $\mathcal{N}at(\mathcal{F},\mathcal{G})$. It is easy to see that the composition $t\circ f$ of natural transformation s of $\mathcal{F}$ into $\mathcal{G}$ and t of $\mathcal{G}$ into $\mathcal{H}$ is a natural transformation of $\mathcal{F}$ into $\mathcal{H}$. If $\mathcal{F}\equiv\mathcal{G}$, then we say that the natural transformation $s:\mathcal{F}\longrightarrow\mathcal{G}$ is a natural transformation of $\mathcal{F}$ into itself.

Now let us briefly explain the notions of monoidal categories [2, 3] adopted for our goal. *A monoidal category* $\mathcal{M}\equiv\mathcal{M}(\otimes,I)$ is in fact a category $\mathcal{M}$ equipped with a monoidal operation (a bifunctor) $\otimes:\mathcal{M}\times\mathcal{M}\longrightarrow\mathcal{M}$, a unit object I, and collections of natural isomorphisms:
(i) an associativity constraint $\psi=\{\psi_{\mathcal{U},\mathcal{V},\mathcal{W}}:(\mathcal{U}\otimes\mathcal{V})\otimes\mathcal{W}\longrightarrow\mathcal{U}\otimes(\mathcal{V}\otimes\mathcal{W})\}$,
(ii) a left unity constraint $l=\{l_{\mathcal{U}}:I\otimes\mathcal{U}\longrightarrow\mathcal{U}\}$

(iii) and a right unity constraint $r = \{r_{\mathcal{U}} : \mathcal{U} \otimes k \longrightarrow \mathcal{U}\}$
such that the following diagrams

$$
\begin{tikzcd}[column sep=small]
 & (\mathcal{U}\otimes\mathcal{V})\otimes(\mathcal{W}\otimes\mathcal{X}) \arrow[dr, "\psi_{\mathcal{U},\mathcal{V},\mathcal{W}\otimes\mathcal{X}}"] & \\
((\mathcal{U}\otimes\mathcal{V})\otimes\mathcal{W})\otimes\mathcal{X} \arrow[ur, "\psi_{\mathcal{U}\otimes\mathcal{V},\mathcal{W},\mathcal{X}}"] \arrow[d, "\psi_{\mathcal{U},\mathcal{V},\mathcal{W}}\otimes id"'] & & \mathcal{U}\otimes(\mathcal{V}\otimes(\mathcal{W}\otimes\mathcal{X})) \\
(\mathcal{U}\otimes(\mathcal{V}\otimes\mathcal{W}))\otimes\mathcal{X} \arrow[rr, "\psi_{\mathcal{U},\mathcal{V}\otimes\mathcal{W},\mathcal{X}}"'] & & \mathcal{U}\otimes((\mathcal{V}\otimes\mathcal{W})\otimes\mathcal{X}) \arrow[u, "id\otimes\psi_{\mathcal{V},\mathcal{W},\mathcal{X}}"']
\end{tikzcd}
\tag{27}
$$

$$
\begin{tikzcd}
(\mathcal{V}\otimes I)\otimes\mathcal{W} \arrow[rr, "\psi_{\mathcal{V},I,\mathcal{W}}"] \arrow[dr, "r_{\mathcal{V}}\otimes id"'] & & \mathcal{V}\otimes(I\otimes\mathcal{W}) \arrow[dl, "id\otimes l_{\mathcal{W}}"] \\
 & \mathcal{V}\otimes\mathcal{W} &
\end{tikzcd}
\tag{28}
$$

commute. It is interesting that in a monoidal category any diagram built from the constraints ψ, l, r, and the identities by composing and tensoring, commutes. This is just the famous Mac Lane's coherence theorem. A monoidal category $\mathcal{M}$ is said to be *strict*, if all natural isomorphisms $\psi_{\mathcal{U},\mathcal{V},\mathcal{W}}, l_{\mathcal{U}}, r_{\mathcal{U}}$ are identity. It is also interesting that every monoidal category is equivalent to certain strict one. This means that we can restrict our attention to strict monoidal categories.

A (left) $*$-operation in a monoidal category $\mathcal{M}$ is a transformation $(-)^*$ of functor $\otimes$ into the opposite functor $\otimes^{op}$ such that

$$
(-)^{**} = id_{\mathcal{M}}, \quad (-)^* \circ \otimes = \otimes^{op} \circ (-)^* \tag{29}
$$

where $\mathcal{U}$ and $\mathcal{V}$ are arbitrary objects of the category $\mathcal{M}$. *A (left) pairing* g in the category $\mathcal{M}$ is a transformation of the functor $(-)^* \otimes -$ into I, where I is a field satisfying some compatibility axioms, see [21, 22]. This means that g is a set $g \equiv \{g_{\mathcal{U}}\}$ of I–valued mappings

$$
g \equiv \{g_{\mathcal{U}} : \mathcal{U}^* \otimes \mathcal{U} \longrightarrow I, \mathcal{U} \in \mathcal{O}b(\mathcal{M})\} \tag{30}
$$

Let $\mathcal{M}$ be a monoidal category equipped with a (left) $*$-operation $(-)^*$ and a (left) pairing g, then such category is said to be a *category with (left) duality* and it is denoted by $\mathcal{M} = \mathcal{M}_{left}(\otimes, \oplus, I, *, g)$. One can introduce a (right) duality structure in the category $\mathcal{M}$ in a similar way. Note that both dualities in $\mathcal{M}$ the right and the left one are in general two independent structures. But it is possible to introduce an additional structure which making these two structures equivalent. Such equivalence can be established by the following set of natural isomorphisms

$$\Psi \equiv \{\Psi_{\mathcal{U}^*,\mathcal{V}} : \mathcal{U}^* \otimes \mathcal{V} \longrightarrow \mathcal{V} \otimes \mathcal{U}^*\}. \tag{31}$$

where

$$\begin{array}{l} \Psi_{\mathcal{U}^*\otimes\mathcal{V}^*,\mathcal{W}} = (\Psi_{\mathcal{U}^*,\mathcal{W}} \otimes id_{\mathcal{V}}) \circ (id_{\mathcal{U}} \otimes \Psi_{\mathcal{V}^*,\mathcal{W}}), \\ \Psi_{\mathcal{U}^*,\mathcal{V}\otimes\mathcal{W}} = (id_{\mathcal{V}} \otimes \Psi_{\mathcal{U}^*,\mathcal{W}}) \circ (\Psi_{\mathcal{U}^*,\mathcal{V}} \otimes id_{\mathcal{W}}), \end{array} \tag{32}$$

for every objects $\mathcal{U}, \mathcal{V}, \mathcal{W}$ in $\mathcal{M}$. These transformations are called a *generalized cross symmetry*, [21]. We can identify the right and left duality in the category equipped with such generalized cross symmetry. The monoidal category equipped with such symmetry is denoted by $\mathcal{M} = \mathcal{M}(\otimes, \oplus, I, *, g, \Psi_{cross})$.

Note that the generalized cross symmetry is not a braid symmetry in general. For the braid symmetry in the category with duality we need additional transformations like

$$\Psi \equiv \{\Psi_{\mathcal{U},\mathcal{V}} : \mathcal{U} \otimes \mathcal{V} \longrightarrow \mathcal{V} \otimes \mathcal{U}\} \tag{33}$$

for arbitrary objects $\mathcal{U}, \mathcal{V}$ in $\mathcal{M}$ and

$$\Psi \equiv \{\Psi_{\mathcal{U}^*,\mathcal{V}^*} : \mathcal{U}^* \otimes \mathcal{V}^* \longrightarrow \mathcal{V}^* \otimes \mathcal{U}^*\} \tag{34}$$

for objects $\mathcal{U}^*, \mathcal{V}^*$ in $\mathcal{M}$. We need also some new commutative diagrams for all these transformations and pairings. In fact a family of natural isomorphisms

$$\Psi \equiv \{\Psi_{\mathcal{U}\otimes\mathcal{W}} : \mathcal{U} \otimes \mathcal{W} \longrightarrow \mathcal{W} \otimes \mathcal{U}\} \tag{35}$$

such that we have the following relations

$$\begin{array}{l} \Psi_{\mathcal{U}\otimes\mathcal{V},\mathcal{W}} = (\Psi_{\mathcal{U},\mathcal{W}} \otimes id_{\mathcal{V}}) \circ (id_{\mathcal{U}} \otimes \Psi_{\mathcal{V},\mathcal{W}}), \\ \Psi_{\mathcal{U},\mathcal{V}\otimes\mathcal{W}} = (id_{\mathcal{V}} \otimes \Psi_{\mathcal{U},\mathcal{W}}) \circ (\Psi_{\mathcal{U},\mathcal{V}} \otimes id_{\mathcal{W}}), \end{array} \tag{36}$$

is said to be *a braiding or a braid symmetry* on $\mathcal{M}$. The monoidal category with unique duality and braid symmetry is said to be *rigid* [9, 10, 11, 12].

This category is denoted by $\mathcal{M} = \mathcal{M}(\otimes, \oplus, I, *, g.\Psi_{braid})$. If in addition we have the relation

$$\Psi^2_{\mathcal{U},\mathcal{V}} = id_{\mathcal{U}\otimes\mathcal{V}}, \tag{37}$$

for every objects $\mathcal{U}, \mathcal{V} \in \mathcal{M}$, then the set $S := \{\Psi_{\mathcal{U},\mathcal{V}}\}$ is said to be a (vector) symmetry or tensor symmetry and the corresponding category $\mathcal{M}$ is called *a symmetric monoidal or tensor category*, see [5, 7].

Let us consider some examples of monoidal categories which can be useful for the study of category symmetry. The most simple example of a monoidal category is provided by the category $\mathcal{V}ect(k)$ of vector spaces over a field k. The monoidal operation in this category is defined by the usual tensor product of vector spaces. Another example is given by the category $\mathcal{V}ect_G(k)$ of G–graded vector spaces, where G is a grading group. In the supersymmetry the grading group is the group of integer Z_2. For anyons we have $G \equiv Z_n$, where $n > 2$, [13].There is a category ${}_{\mathcal{B}}\mathcal{M}$ of all left $\mathcal{B}$-modules, where $\mathcal{B}$ is an unital and associative algebra. Observe that the usual tensor product $\mathcal{U} \otimes \mathcal{V}$ of two left $\mathcal{B}$–modules $\mathcal{U}$ and $\mathcal{V}$ is not a left $\mathcal{B}$-module but a left $\mathcal{B} \otimes \mathcal{B}$-module! Hence this category is not a monoidal category. But it is easy to see that in the particular case when $\mathcal{B}$ is a bialgebra, i.e. we have a comultiplication $\triangle : \mathcal{B} \longrightarrow \mathcal{B} \otimes \mathcal{B}$ in $\mathcal{B}$, the category ${}_{\mathcal{B}}\mathcal{M}$ is monoidal. For instance there is the category $\mathcal{R}_G$ of finite dimensional representations of compact matrix quantum group G, [26]. There is also a category of Hopf modules or crossed modules [22]. Observe that there is also a category $\mathcal{M}^{\mathcal{H}}$ of right $\mathcal{H}$–comodules, where $\mathcal{H}$ is a Hopf algebra. The monoidal operation in $\mathcal{M}^{\mathcal{H}}$ is given as the following tensor product of $\mathcal{H}$–comodules

$$\rho_{\mathcal{U}\otimes\mathcal{V}} = (id \otimes m_{\mathcal{H}}) \circ (id \otimes \tau \otimes id) \circ (\rho_{\mathcal{U}} \otimes \rho_{\mathcal{V}}), \tag{38}$$

where $\tau : \mathcal{U} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{U}$ is the twist, $m_{\mathcal{H}} : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$ is the multiplication in H.

As an example for a category with duality we can give a category ${}_{\mathcal{B}}\mathcal{M}$ of left $\mathcal{B}$–modules. In this case the monoidal operation corresponds to the tensor product of representations, the $*$-operation corresponds to the contragradient representation and the generalized cross symmetry corresponds to the intertwiner between an arbitrary representation and its contragradient. Hence it is also called *a statistics operator*. Note that if there is a bialgebra $\mathcal{B}$ such that the category $\mathcal{M}$ is equivalent to the category of left modules (representations) over $\mathcal{B}$, then the bialgebra is said to be *a generalized symmetry*

or *(bi-)algebra symmetry* for the given physical system. One can describe states in the quantum Hall effect as a result of symmetry in such generalized sense [28]. The symmetry algebra for Klein–Gordon equation on quantum Minkowski space is considered in [29].

Note that the category of representations of the so–called weak Hopf algebra is rigid [27]. Also the category of quantum compact matrix groups of Woronowicz is rigid [26, 19]. For a coquasitriangular Hopf algebra H with a coquasitriangular structure $\langle -,-\rangle : H\otimes H \longrightarrow I$ we obtain the category $\mathcal{M}^H$ of right H-comodules which is also braided monoidal [1]. The braid symmetry $\Psi \equiv \{\Psi_{\mathcal{U},\mathcal{V}} : \mathcal{U}\otimes\mathcal{V} \longrightarrow \mathcal{V}\otimes\mathcal{U}; \mathcal{U},\mathcal{V}\in Ob\mathcal{M}\}$ in $\mathcal{M}$ is defined by the equation

$$\Psi_{\mathcal{U},\mathcal{V}}(u\otimes v) = \Sigma\langle v_1, u_1\rangle\ v_0\otimes u_0, \tag{39}$$

where $\rho(u) = \Sigma u_0\otimes u_1 \in \mathcal{U}\otimes H$, and $\rho(v) = \Sigma v_0\otimes v_1 \in \mathcal{V}\otimes H$ for every $u\in\mathcal{U}, v\in\mathcal{V}$.

Let G be an arbitrary group, then the group algebra $H := \mathbb{C}G$ is a Hopf algebra for which the comultiplication, the counit, and the antypode are given by the formulae

$$\triangle(g) := g\otimes g, \quad \eta(g) := 1, \quad S(g) := g^{-1} \quad \text{for } g\in G.$$

respectively. If $H \equiv \mathbb{C}G$, where G is an Abelian group, then the coquasitriangular structure on H is given as a bicharacter on G [36]. Note that for Abelian groups we use the additive notation. A mapping $\epsilon : G\times G \longrightarrow \mathbb{C}\backslash\{0\}$ is said to be a *bicharacter* on G if and only if we have the following relations

$$\epsilon(\alpha,\beta+\gamma) = \epsilon(\alpha,\beta)\epsilon(\alpha,\gamma), \quad \epsilon(\alpha+\beta,\gamma) = \epsilon(\alpha,\gamma)\epsilon(\beta,\gamma) \tag{40}$$

for $\alpha,\beta,\gamma\in G$. If in addition

$$\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = 1, \tag{41}$$

for $\alpha,\beta\in G$, then ϵ is said to be a *normalized bicharacter* or a commutation factor on G [35]. The category $\mathcal{M}^H$ of right comodules, where $H := \mathbb{C}G$ for certain Abelian group G and $\langle -,-\rangle \equiv \epsilon(-,-)$ is a bicharacter like above is denoted by $\mathcal{M}(G,\epsilon)$. Note that if $\mathcal{U}$ is a H-comodule, where $H = \mathbb{C}G$, then $\mathcal{U}$ is a G-graded vector space, i.e $\mathcal{U} = \bigoplus_{\alpha\in G} \mathcal{U}_\alpha$. This means that a coaction of $H := \mathbb{C}G$ on $\mathcal{U}$ is equivalent to G-gradation of $\mathcal{U}$.

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